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Lower Bounds on the Squashed Entanglement for Multi-Party System

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Abstract Squashed entanglement is a promising entanglement measure that can be generalized to multipartite case, and it has all of the desirable properties for a good entanglement measure. In this paper we present computable lower bounds to evaluate the multipartite squashed entanglement. We also derive some inequalities relating the squashed entanglement to other entanglement measures.

Keywords Entanglement measure · Squashed entanglement · Lower bound

Entanglement has been recognized as a key resource and ingredient in the field of quantum information and computation science. As a result, a remarkable research effort has been devoted to characterizing and quantifying it (see, e.g., [1, 2] and references therein). Despite a large number of profound results obtained in this field, e.g., [3–32], there is still no general solution to the simplest case, namely the two partite case. It is usually accepted that the following two axioms [20] are satisfied for an appropriate entanglement measure. One natural axiom is that an entanglement measure should not increase under local operations and classical communication [8]. The other is that every entanglement measure should vanish on the set of separable quantum states. Some other useful but not necessary properties require the entanglement measures should be convex, additive, and a continuous function in the state. The issue of entanglement measure for multipartite states poses an even greater challenge [33, 34], and most of existing entanglement measures are constructed for bi-partite state except that the quantum relative-entropy of entanglement [5] and squashed entanglement [22] can be generalized to multipartite case. Among the existing bi-partite entanglement measures, additivity only holds for squashed entanglement and logarithmic negativity [21] and is conjectured to hold for entanglement of formation, but the quantum relative-entropy of entanglement is nonadditive [35]. Squashed entanglement was introduced by [36] and then independently by Christandl and Winter [22], who showed that it is monotone, and proved

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its additivity. It has all of the desirable properties for a good entanglement measure: it is convex, asymptotically continuous, additive on tensor products and superadditive in general. It is upper bounded by entanglement cost, lower bounded by distillable entanglement. Very recently, the squashed entanglement was extended to multipartite case by Yang et al. [37] and similar ideas have also been developed independently in [38]. Furthermore, in a recent paper [39], the squashed entanglement is given the operational meaning with the aid of conditional mutual information. Thus the squashed entanglement is a promising candidate among the different kinds of entanglement measures. However, it is still very difficult to compute the squashed entanglement and no analytic formula exists even for bipartite states. In fact, it is usually not easy to evaluate entanglement measures. Entanglement of formation is efficiently computable only two-qubit system [6]. Other measures are usually computable for states with high symmetries, such as Werner states, isotropic states, or the family of "iso-Werner" states, and squashed entanglement can only be evaluated for so called special flower states [40].

In this paper our aim is to explore a computable lower bound to evaluate the multipartite squashed entanglement. Firstly we briefly review the definition of multipartite q-squashed entanglement introduced in [37]. Before describing the details of multipartite squashed entanglement, it is necessary to recall the definition of multipartite mutual information. In this paper we will adopt the function $I(A_1 : A_2 : \ldots : A_n) = S(A_1) + S(A_2) + \ldots + S(A_n) - S(A_1A_2\ldots A_n)$ as a multipartite mutual information, where $S(X)$ is the von Neumann entropy of system X. This version of multipartite mutual information has an interesting feature: it can be represented as a sum of bipartite mutual informations: $I(A_1 : A_2 : \ldots : A_n) = I(A_1 : A_2) + I(A_3 : A_1A_2) + I(A_4 : A_1A_2A_3) + \ldots + I(A_n : A_1A_2\ldots A_{n-1})$. Analogous to the definition of bipartite conditional mutual information $I(A : B|E) = S(AE) + S(BE) - S(ABE) - S(E)$, we can also define the multipartite conditional mutual information $I(A_1 : A_2 : \ldots : A_N|E)$. For the N-party state $\rho_{A_1\ldots A_N}$, the multipartite q-squashed entanglement is defined as

$$E_{sq}^{q}\left(\rho_{A_1\ldots A_N}\right) = \inf I\left(A_1 : A_2 : \ldots : A_N|E\right), \tag{1}$$

where the infimum is taken over states $\sigma_{A_1\ldots A_N,E}$, that are extensions of $\rho_{A_1\ldots A_N}$, i.e. $\mathrm{Tr}_E\sigma = \rho$. If the extension states $\sigma_{A_1,\ldots,A_N,E}$ takes the form $\sum_i p_i\rho^i_{A_1,\ldots,A_N} \otimes |i\rangle_E\langle i|$, we call it c-squashed entanglement. Here, we denote q-squashed entanglement and c-squashed entanglement both as $E_{sq}(\rho_{A_1\ldots A_N})$ due to our derivation is irrelevant to the form of the extension states. We begin by considering tri-partite state and later generalize the results to the case of multi-party subsystem. Notice that $I(A_1 : A_2 : \ldots : A_N|E)$ can be represented as the sum of the following terms:

$$\begin{aligned} I\left(A_1 : A_2 : \ldots : A_N|E\right) &= I\left(A_1 : A_2|E\right) + I\left(A_3 : A_1A_2|E\right) \\ &\quad + I\left(A_4 : A_1A_2A_3|E\right) + \cdots + I\left(A_N : A_1A_2A_3\ldots A_{N-1}|E\right). \end{aligned} \tag{2}$$

Now we can prove the following:

Lemma 1 *For any tri-partite state $\rho_{A_1A_2A_3}$, we have*

$$E_{sq}\left(\rho_{A_1:A_2:A_3}\right) \geq \max\left\{C - S\left(A_1A_2\right), C - S\left(A_1A_3\right), C - S\left(A_2A_3\right)\right\}, \tag{3}$$

where $C = \sum_{i=1}^{3} S(A_i) - 2S(A_1A_2A_3)$.

Proof Suppose that E is an optimum extension for system $A_1A_2A_3$ satisfying $E_{sq}(\rho_{A_1:A_2:A_3}) = I(A_1 : A_2 : A_3|E)$. Then

$$\begin{aligned} & E_{sq}\left(\rho_{A_1:A_2:A_3}\right) - 2E_{sq}\left(\rho_{A_1:A_2}\right) - 2E_{sq}\left(\rho_{A_1A_2:A_3}\right) \\ & \quad \geq I\left(A_1 : A_2 : A_3|E\right) - I\left(A_1 : A_2|E\right) - I\left(A_1A_2 : A_3|E\right) = 0. \end{aligned} \tag{4}$$

Thus we have $E_{sq}(\rho_{A_1:A_2:A_3}) \geq 2E_{sq}(\rho_{A_1:A_2}) + 2E_{sq}(\rho_{A_1A_2:A_3})$. Employing a lower bound of the bi-partite squashed entanglement presented in [22], thus we obtain: $E_{sq}(\rho_{A_1:A_2:A_3}) \geq \sum_{i=1}^{3} S(A_i) - S(A_1A_2) - 2S(A_1A_2A_3)$. If we permute the indices cyclically we get three inequalities and obtain the sharpest bound. This ends the proof. □

It should be noted that the constant 2 in (4) is due to the difference of the definition between bipartite squashed entanglement and multipartite squashed entanglement.The measures we propose in the case of two parties reduces to twice the original squashed entanglement.

Corollary 1 *For any tri-partite state* $\rho_{A_1A_2A_3}$, *we have*

$$E_{sq}\left(\rho_{A_1:A_2:A_3}\right) \geq 2E_{sq}\left(\rho_{A_1:A_2}\right) + 2E_{sq}\left(\rho_{A_2:A_3}\right) + 2E_{sq}\left(\rho_{A_1:A_3}\right). \tag{5}$$

Proof Notice that the monogamy inequality of bi-partite squashed entanglement [41], i.e., $E_{sq}(\rho_{A:BC}) \geq E_{sq}(\rho_{A:B}) + E_{sq}(\rho_{A:C})$, the proof is obtained immediately. □

By taking the average over all combinations of two parties in (3) we get the following corollary:

Corollary 2 *For any tri-partite states* $\rho_{A_1A_2A_3}$, *we have*

$$\begin{aligned} E_{sq}\left(\rho_{A_1:A_2:A_3}\right) \geq{} & S\left(A_1\right) + S\left(A_2\right) + S\left(A_3\right) - \frac{1}{3}\left[S\left(A_1A_2\right) + S\left(A_2A_3\right) + S\left(A_1A_3\right)\right] \\ & - 2S\left(A_1A_2A_3\right). \end{aligned} \tag{6}$$

Equation (3) *and* (6) *provide computable lower bounds to evaluate the tri-partite squashed entanglement. Using an inequality presented in* [42], *we can also relate the relative-entropy of entanglement to the squashed entanglement measure. For tri-partite pure state we have* $E_{sq}(\rho_{A_1:A_2:A_3}) = S(A_1) + S(A_2) + S(A_3)$. *Employing the inequality* (12) *in* [42] *an immediate corollary is as follows*:

Corollary 3

$$\begin{aligned} & \frac{3}{2}E_{RE}\left(\rho_{A_1:A_2:A_3}\right) \leq E_{sq}\left(\rho_{A_1:A_2:A_3}\right) \\ & \qquad \leq 3E_{RE}\left(\rho_{A_1:A_2:A_3}\right) - E_{RE}\left(\rho_{A_1:A_2}\right) - E_{RE}\left(\rho_{A_1:A_3}\right) - E_{RE}\left(\rho_{A_2:A_3}\right) \end{aligned} \tag{7}$$

for any pure tri-partite state $\rho_{A_1A_2A_3}$.

Furthermore, we can derive an inequality relating the conditional entanglement of mutual information with the squashed entanglement. Conditional entanglement of mutual information is a new entanglement measure introduced in [30, 31]. Remarkably, it is additive and has an operational meaning and can straightforwardly be generalized to multipartite cases. Conditional entanglement of mutual information is defined as follows:

Definition Let ρ_{AB} be a mixed state on a bipartite Hilbert space $\mathrm{H}_A \otimes \mathrm{H}_B$. The conditional entanglement of mutual information for ρ_{AB} is defined as

$$C_I(\rho_{AB}) = \inf \frac{1}{2}\left\{I\left(AA' : BB'\right) - I\left(A' : B'\right)\right\}, \tag{8}$$

where the infimum is taken over all extensions of ρ_{AB}, i.e., over all states satisfying the equation $\mathrm{Tr}_{A'B'}\rho_{AA'BB'} = \rho_{AB}$, and the factor $1/2$ is to make it equal to the entanglement of formation for the pure state case. Yang et al. [30, 31] have proved that C_I satisfied all the desired property of a good entanglement measure and it is easily generalized to the multipartite case. For multipartite mixed state $\rho_{A_1A_2\ldots A_n}$, $C_I(\rho_{A_1\ldots A_n}) = \inf\{I_n(A_1A_1' : \ldots : A_nA_n') - I_n(A_1' : \ldots : A_n')\}$, where $I_n = \sum_i S(A_i) - S(A_1 \ldots A_n)$. Now we present our result which is the following lemma.

Lemma 2 *For any tri-partite state $\rho_{A_1A_2A_3}$, we have*

$$\begin{aligned} C_I\left(\rho_{A_1:A_2:A_3}\right) \geq \max\Big\{&2C_I\left(\rho_{A_1:A_2}\right) + 2E_{sq}\left(\rho_{A_1A_2:A_3}\right), 2C_I\left(\rho_{A_1:A_3}\right) + 2E_{sq}\left(\rho_{A_1A_3:A_2}\right), \\ &2C_I\left(\rho_{A_2:A_3}\right) + 2E_{sq}\left(\rho_{A_2A_3:A_1}\right)\Big\}. \end{aligned} \tag{9}$$

Proof Suppose that $A_1'A_2'A_3'$ is a minimum extension for system $A_1A_2A_3$ satisfying $C_I(\rho_{A_1:A_2:A_3}) = I_3(A_1A_1' : A_2A_2' : A_3A_3') - I_3(A_1' : A_2' : A_3')$. Then

$$\begin{aligned} &C_I\left(\rho_{A_1:A_2:A_3}\right) - 2C_I\left(\rho_{A_1:A_2}\right) - 2E_{sq}\left(\rho_{A_1A_2:A_3}\right) \\ &\quad\geq C_I\left(\rho_{A_1:A_2:A_3}\right) - 2C_I\left(\rho_{A_1:A_2}\right) - I\left(A_1A_2 : A_3|A_3'\right) \\ &\quad\geq C_I\left(\rho_{A_1:A_2:A_3}\right) - 2C_I\left(\rho_{A_1:A_2}\right) - I\left(A_1A_1'A_2A_2' : A_3|A_3'\right) \\ &\quad\geq S\left(A_1A_1'\right) + S\left(A_2A_2'\right) + S\left(A_3A_3'\right) - S\left(A_1A_1'A_2A_2'A_3A_3'\right) \\ &\qquad - S\left(A_1'\right) - S\left(A_2'\right) - S\left(A_3'\right) + S\left(A_1'A_2'A_3'\right) \\ &\qquad - S\left(A_1A_1'\right) - S\left(A_2A_2'\right) + S\left(A_1A_1'A_2A_2'\right) \\ &\qquad + S\left(A_1'\right) + S\left(A_2'\right) - S\left(A_1'A_2'\right) - I\left(A_1A_1'A_2A_2' : A_3|A_3'\right) \\ &\quad= S\left(A_1A_1'A_2A_2'\right) + S\left(A_1'A_2'A_3'\right) \\ &\qquad - S\left(A_1'A_2'\right) - S\left(A_1A_1'A_2A_2'A_3'\right) \geq 0. \end{aligned} \tag{10}$$

The last inequality is due to strong subadditivity of the von Neumann entropy. Analogously we can prove the other two inequalities. □

Next we generalize our lower bounds on the squashed entanglement to the N-partite case. Using the similar procedure as proving Lemma 1, we obtain the following general result:

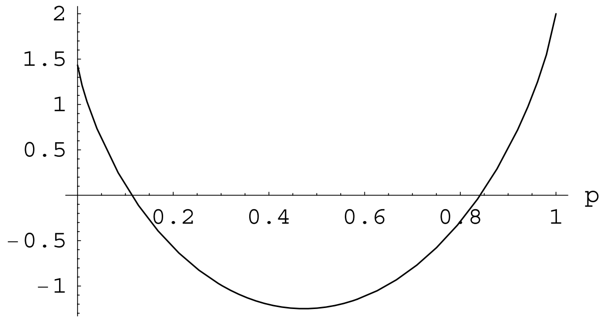


Fig. 1 Plot of the lower bound of the squashed entanglement for the mixed state $\rho(p)$

Lemma 3 *For any N-partite state $\rho_{A_1A_2\ldots A_N}$, we have*

$$E_{sq}\left(\rho_{A_1:A_2:\ldots:A_N}\right) \geq \sum_{i=1,2,\ldots,N}^{N} S\left(A_i\right) - \sum_{M=2,\ldots,N-1} \frac{1}{\binom{N}{M}} \sum_{i_1<\ldots<i_M=1,2,\ldots,N}^{N} S\left(A_{i_1}\ldots A_{i_M}\right) - 2S\left(A_1\ldots A_N\right). \tag{11}$$

Finally, we show an inequality of the multipartite squashed entanglement analogous to the monogamy inequality for the bi-partite case.

Lemma 4 *For any multipartite state $\rho_{A_1A_2\ldots A_N}$*

$$E_{sq}\left(\rho_{A_1:A_2:\ldots:(A_{N-1}A_N)}\right) \geq E_{sq}\left(\rho_{A_1:A_2:\ldots:A_{N-1}}\right) + E_{sq}\left(\rho_{(A_1A_2\ldots A_{N-2}):A_N}\right). \tag{12}$$

Proof Suppose that E is a minimum extension for state $\rho_{A_1A_2\ldots A_N}$, then

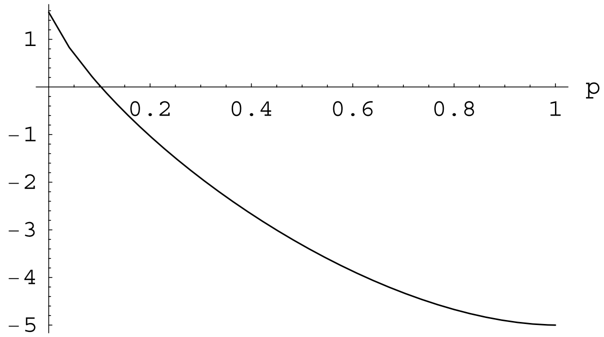
$$\begin{aligned} E_{sq}\left(\rho_{A_1:A_2:\ldots:(A_{N-1}A_N)}\right) &= I\left(A_1 : A_2 : \ldots : (A_{N-1}A_N)\,|E\right) \\ &= I\left(A_1 : A_2 : \ldots : A_{N-1}|E\right) + I\left((A_1A_2\ldots A_{N-2}) : A_N|A_{N-1}E\right) \\ &\geq E_{sq}\left(\rho_{A_1:A_2:\ldots:A_{N-1}}\right) + E_{sq}\left(\rho_{(A_1A_2\ldots A_{N-2}):A_N}\right). \end{aligned} \tag{13}$$

□

Below we give some examples to show the application of (11).

Example 1 Consider a family of mixed 4-qubit state $\rho(p) = p|GHZ\rangle\langle GHZ| + (1-p)\times |W\rangle\langle W|$, where $|GHZ\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$, and $|W\rangle = \frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)$. In order to evaluate the multipartite entanglement of $\rho(p)$, we plot the lower bound of the squashed entanglement as a function of p in Fig. 1. We find the lower bound for $0 \leq p < 0.113$ and $0.842 < p \leq 1$ is positive, which shows that $\rho(p)$ is an entangled state in these cases. It should be noted that the analytic expression of the 3-tangle for the 3-qubit state $\rho(p)$ have been obtained in [43] recently, and the 3-tangle can be used as an entanglement measure for the genuine 3-party entanglement. However, their results only restricted to the 3-qubit state and it is not obviously to generalize the 3-tangle to the multipartite case. In contrast, our lower bound can be used to evaluate the squashed entanglement for arbitrary party systems.

Fig. 2 Plot of the lower bound of the squashed entanglement for the mixed state $\rho_W(p)$



Example 2 Consider a class of generalized Werner states [44, 45] for $2 \otimes 2 \otimes 2$ systems: $\rho_W(p) = \frac{p}{8} I \otimes I \otimes I + (1-p)|\psi\rangle\langle\psi|$, where $|\psi\rangle = \frac{1}{\sqrt{6}}(2|110\rangle - |101\rangle - |011\rangle)$.[1] The tripartite mixed state $\rho_W(p)$ are invariant under $\rho_W \to \int dU U \otimes U \otimes U \rho_W U^\dagger \otimes U^\dagger \otimes U^\dagger$ and can be regarded as generalized tripartite Werner states. Now we employ the lower bound to evaluate the squashed entanglement of $\rho_W(p)$. The lower bound is plotted in Fig. 2. We can still get a positive lower bound for $0 \le p < 0.103$.

Our results provide computable lower bounds on the multipartite squashed entanglement for the first time, which allow us to evaluate the multipartite squashed entanglement for a wide class of mixed states. These bounds also help us to judge whether a general mixed multipartite state is entangled or not, and some useful results can be obtained in some cases. We also relate the squashed entanglement to the other entanglement measure, such as quantum relative-entropy of entanglement, and conditional entanglement of mutual information. An interesting question remained is to derive a tighter lower bound of the multipartite squashed entanglement or the upper bound of the squashed entanglement for the bi-partite and multipartite case.

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[1] Here, we choose $|\psi\rangle = \frac{1}{\sqrt{6}}(2|110\rangle - |101\rangle - |011\rangle)$ which is one of the basis of the decoherence-free subspace, i.e., it is invariant under the collective noise $U \otimes U \otimes U$.

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